

THE DECAY RATES OF TRAVELING WAVES FOR A CLASS OF NONLOCAL EVOLUTION EQUATIONS

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ABSTRACT. We obtain the precise decay rates of traveling wave for a class of nonlocal evolution equations arising in the theory of phase transitions. We also investigate the spectrum of the operator obtained by linearizing at such a traveling wave. The detailed description of the spectrum is established.

1. INTRODUCTION

In this paper, we are concerned with a class of nonlocal evolution equations of the form

$$(1.1) \quad \frac{\partial u(x, t)}{\partial t} = d \frac{\partial^2 u(x, t)}{\partial x^2} + f(u(x, t), (J * u)(x, t))$$

for $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$. Here $d \geq 0$ is a constant, $(J * u)(x, t) := \int_{\mathbb{R}} J(x - y)u(y, t)dy$, f and J are sufficiently smooth functions. Depending upon the constant d and the nonlinearity f involved, equation (1.1) may model the spatio-temporal development of various populations or epidemics (see the surveys and references cited therein). Similar equations have been also derived and studied from the point of view of certain continuum limits in the dynamic Ising models (see [2], [3], [4], [5], [6] and references therein). Equation (1.1) has received much attention recently, the possible interest of such an equation lies in the fact that much more general types of interactions in the medium can be account for. The existence as well as the uniqueness of a traveling wave solution for integro-differential equations (1.1) have been of great interest, both from a mathematical standpoint and for their applications. Indeed, our study of (1.1) is motivated by the following traveling wave problems.

A. Family of neurons

$$\frac{\partial u}{\partial t} = -u + \int_{\mathbb{R}} J(x - y)S(u(y, t))dy,$$

where $m(u) := S(u) - u$ satisfies $m'(0) < 0$, $m'(1) < 0$. J is a smooth kernel satisfying

$$(1.2) \quad J \geq 0 \text{ on } \mathbb{R}, \quad \int_{\mathbb{R}} J = 1.$$

B. Ising model

$$\frac{\partial u}{\partial t} = \tanh \beta (J * u + h) - u,$$

where $\beta > 1$ is inverse temperature and h is a constant. J is a smooth kernel supported in $[-1, 1]$ satisfying (1.2).

C. Phase transition

$$\frac{\partial u}{\partial t} = \varepsilon [J * u - u] + g(u),$$

where $g(u)$ is a bistable function, $\lambda > 0$.

Throughout this paper, we make the following hypotheses.

(H1) $d + |c| \neq 0$.

(H1) $J \in C(\mathbb{R})$ is even, nonnegative such that

$$\int_{\mathbb{R}} J(s) ds = 1 \quad \text{and} \quad \int_{\mathbb{R}} J(s) e^{\rho s} ds < +\infty \quad \text{for any } \rho \in \mathbb{R}.$$

(H2) $f \in C^{2,\alpha}(\mathbb{R} \times \mathbb{R})$ and $f(-1, -1) = f(1, 1) = f(q, q) = 0$, where $-1 < q < 1$.

(H3) $\partial_s f(r, s) > 0$ for any fixed $(r, s) \in [-1, 1] \times [-1, 1]$.

(H4) $\partial_r f(\pm 1, \pm 1) < 0$ and $\partial_r f(\pm 1, \pm 1) < -\partial_s f(\pm 1, \pm 1)$.

(H5) $\bar{f}(\cdot) = f(\cdot, \cdot)$ is bistable, i.e. \bar{f} has exactly three zeros ± 1 and q . There exists an interval $[l, l'] \subset (-1, 1)$ such that $q \in [l, l']$, $\bar{f}'(s) \geq 0$ for any $s \in [l, l']$ and $\bar{f}'(s) \leq 0$ for any $s \in [-1, 1] \setminus [l, l']$.

Under conditions (H1)-(H5), it is well known that equation (1.1) possesses a unique monotone traveling wave solution connecting the equilibria ± 1 (i.e solutions of the form $u(x, t) = U(x + ct)$ for some velocity c , $\lim_{\xi \rightarrow \pm\infty} U(\xi) = \pm 1$ with $\xi = x + ct$.) However, the precise rates at which U approaches the two homogeneous equilibria ± 1 are still lacking. In this paper, we address this issue. Our main goal is to obtain the exact decay rates of traveling wave of (1.1) as $\xi \rightarrow \pm\infty$. With the right rates of convergence, we can easily establish the uniqueness of the traveling wave. Recently, a spectral analysis of traveling waves of (1.1) was made in [3]. The authors considered the operator obtained by linearizing (1.1) at U in $C_0(\mathbb{R})$, the space of continuous functions which vanish at infinity. They show that the operator has spectrum in the left half plane, bounded away from the imaginary axis except for an algebraically simple eigenvalue at zero. This fact is of crucial importance, which not only implies the exponential asymptotic stability of traveling waves but also leads to the description of dynamics of the codimension-one invariant stable manifolds. Here the codimension-one invariant stable manifolds are transverse to the one-dimensional manifold formed by the translates of the traveling wave. Based on our study of asymptotical behavior of traveling waves, we are able to obtain a detailed description of the spectrum of the operator in the underlying L^p space ($1 \leq p \leq \infty$).

The paper is organized as follows: In section 2, we investigate the exponential decay rates of the traveling wave and prove its uniqueness. In section 3, we study the spectrum of the operator obtained by linearizing (1.1) at the traveling wave. Some of the results needed in the spectral analysis can be obtained by using arguments similar to those in [15], and therefore we summarize these results in the Appendix with sketched proofs that are necessary for our purposes.

2. DECAY RATES OF TRAVELING WAVES

In this section, we study the asymptotical behavior of traveling wave $(c, U) \in \mathbb{R} \times C^2(\mathbb{R})$ which satisfies

$$(2.1) \quad \begin{cases} cU' = dU'' + f(U, J * U) & \text{on } \mathbb{R}, \\ \lim_{\xi \rightarrow \pm\infty} U(\xi) = \pm 1, \quad U' > 0 & \text{on } \mathbb{R}. \end{cases}$$

We show that the behavior of the traveling wave near $\pm\infty$ is governed by exponentials. Moreover, we determine the exact exponential decay rates of U as $\xi \rightarrow \pm\infty$. For our purpose, we shall adapt the Fourier transform techniques presented in [15] and [18] (see also [24] and [25]). By differentiating equation (2.1) with respect to ξ , we obtain

$$(2.2) \quad c(U')' = d(U')'' + f_r(U, J * U)U' + f_s(U, J * U)J * U'.$$

From (2.1)

$$(2.3) \quad \lim_{\xi \rightarrow \pm\infty} f_r(U, J * U) = f_r(\pm 1, \pm 1), \quad \lim_{\xi \rightarrow \pm\infty} f_s(U, J * U) = f_s(\pm 1, \pm 1).$$

Motivated by (2.2) and (2.3), we consider the linear operator $L : D(L) \subset L^p(\mathbb{R}, \mathbb{C}) \rightarrow L^p(\mathbb{R}, \mathbb{C})$ defined by

$$(2.4) \quad Lv := dv'' - cv' + a(\xi)v + b(\xi)J * v, \quad \xi \in \mathbb{R},$$

where $D(L) := \{v \in L^p | v', dv'' \in L^p\}$, $a, b \in C_b(\mathbb{R}, \mathbb{R})$, and $1 \leq p \leq \infty$.

A special case occurs if both a and b are constants, we define $L_0 : D(L_0) \subset L^p(\mathbb{R}, \mathbb{C}) \rightarrow L^p(\mathbb{R}, \mathbb{C})$ by

$$(2.5) \quad L_0v := dv'' - cv' + av + bJ * v, \quad \xi \in \mathbb{R},$$

where $D(L_0) := D(L)$

In what follows, when convenient, $f_r(\pm 1, \pm 1), f_s(\pm 1, \pm 1)$ are denoted by a^\pm and b^\pm , respectively.

Let $\Delta_0(z) : \mathbb{C} \rightarrow \mathbb{C}$ be the characteristic function associated with L_0 , defined by

$$\Delta_0(z) = dz^2 - cz + a + b \int_{\mathbb{R}} J(s)e^{-zs} ds.$$

In an attempt to solve the inhomogeneous equations

$$(2.6) \quad L_0v = h, \quad h \in L^p,$$

we may formally take the Fourier transform to obtain

$$\Delta_0(i\eta)\widehat{v}(\eta) = \widehat{h}(\eta), \quad \eta \in \mathbb{R},$$

where $\widehat{g}(z) = (2\pi)^{-1} \int_{\mathbb{R}} g(s)e^{-izs} ds$, $i = \sqrt{-1}$ and $z \in \mathbb{C}$. Note that $\Delta_0^{-1}(i\eta) = O(|\eta|^{-1})$. Therefore, we can take the inverse transform of $\Delta_0^{-1}(i\eta)$ to obtain solution v provided $\Delta_0(i\eta) \neq 0$ for any $\eta \in \mathbb{R}$.

Definition 2.1. *The operator L_0 is called hyperbolic if $\Delta_0(i\eta) \neq 0$ for any $\eta \in \mathbb{R}$.*

In what follows, for a given complex number $z \in \mathbb{C}$, we shall always denote its real part and imaginary part by $\Re z$ and $\Im z$, respectively. The following Lemma ensures the existence of $\Delta_0^{-1}(i\eta)$ under suitable conditions.

Lemma 2.1. *Suppose that $a < 0, b > 0$ and $b < -a$. Then*

(a) *The equation $\Delta_0(z) = 0$ has precisely two real solutions $\lambda^s < 0 < \lambda^u$.*

(b) *The zeros of $\Delta_0(z)$ in the vertical strip $\{z \in \mathbb{C} | \lambda^s \leq \Re z \leq \lambda^u\}$ are λ^s and λ^u . In addition, in each vertical strip $|\Re z| \leq K$, there lie only finite number of zeros of $\Delta_0(z)$.*

Proof. Set $N(z) = b \int_{\mathbb{R}} J(s) e^{-zs} ds$ and $D(z) = cz - dz^2 - a$. Then z is a zero of $\Delta_0(z)$ if and only if z is a solution to the equation $N(z) = D(z)$. Due to the assumption, $D(0) - N(0) > 0$. We start with the case that z takes on real values, note that $N(z)$ is a positive convex even function of z with $\frac{\partial^2 N}{\partial z^2} > 0$ for any $z \neq 0$. Consequently, there exists two real roots of $N(z) = D(z)$, denoted by λ^s and λ^u with $\lambda^s < 0 < \lambda^u$. In addition, it is easy to see that $N(z) < D(z)$ if $z \in (\lambda^+, \lambda^u)$, whereas, $N(z) > D(z)$ if $z \in \mathbb{R} \setminus (\lambda^+, \lambda^u)$. This confirms the part (a). Next let $z \in \mathbb{C}$. Note that

$$(2.7) \quad |N(z)| \leq \int_{\mathbb{R}} J(s) e^{-\Re z s} ds < \Re D(z), \quad \forall z \in \{z \in \mathbb{C} | \lambda^s < \Re z < \lambda^u\}.$$

Moreover, we observe that

$$\Re D(\lambda^s + i\mu) \geq \int_{\mathbb{R}} J(t) e^{-\lambda^s t} dt > \int_{\mathbb{R}} J(t) e^{-\lambda^s t} \cos \mu t dt = \Re N(\lambda^s + i\mu)$$

and

$$\Re D(\lambda^u + i\mu) \geq \int_{\mathbb{R}} J(t) e^{-\lambda^u t} dt > \int_{\mathbb{R}} J(t) e^{-\lambda^u t} \cos \mu t dt = \Re N(\lambda^u + i\mu)$$

for any $\mu \neq 0$. This yields the first part of (b). Due to the first inequality in (2.7), $|N(z)|$ is bounded in the vertical strip $|\Re z| \leq K$, $K > 0$. Clearly, when restricted to such a strip, the solution set of $D(z) = N(z)$ is bounded. Since $\Delta_0(z)$ is an entire function over \mathbb{C} , there are only finitely many roots of $\Delta_0(z)$ in such a strip. \square

Lemma 2.2. *Assume that the operators L_0 defined by (2.5) is hyperbolic. Then for each $1 \leq p \leq \infty$, $L_0 : D(L_0) \rightarrow L^p$ is an isomorphisms. The inverses is given by convolution*

$$(L_0^{-1}h)(\xi) = (G_0 * h)(\xi) = \int_{\mathbb{R}} G_0(\xi - \eta) h(\eta) d\eta$$

with the functions G_0 , which enjoy the estimate

$$(2.8) \quad |G_0(\xi)| \leq C e^{-\alpha|\xi|}, \quad \xi \in \mathbb{R}$$

for some positive constants C and α . In particular, for each $h \in L^p$, $u = L_0^{-1}h$ is the unique solution to the inhomogeneous equation (2.6).

Proof. Set

$$(2.9) \quad G_0(\xi) = \int_{\mathbb{R}} e^{i\xi\eta} \Delta_0^{-1}(i\eta) d\eta, \quad \xi \in \mathbb{R}.$$

By a similar argument used in [15], we may interpret G_0 as a tempered distribution and show that

$$(2.10) \quad dG_0''(\xi) - cG_0'(\xi) + aG_0(\xi) + bJ * G_0(\xi) = \delta(\xi),$$

where δ denotes the delta function distribution. Therefore, when $d = 0$, as a function, G_0 is absolutely continuous for all $\xi \neq 0$ and satisfies

$$cG_0'(\xi) = aG_0(\xi) + bJ * G_0(\xi) \quad \text{almost everywhere.}$$

Furthermore, the function G_0 possess left- and right-hand limits $G_0(0-)$ and $G_0(0+)$ at $\xi = 0$, and there is a jump discontinuity

$$G_0(0+) - G_0(0-) = 1.$$

If $d > 0$, then G_0 is absolutely continuous for all ξ and G_0' is discontinuous at $\xi = 0$.

We now show that the function G_0 decays exponentially at $\pm\infty$. First observe that

$$\left| \int_{\mathbb{R}} J(s) e^{-zs} ds \right| \rightarrow 0, \quad |\Im z| \rightarrow \infty$$

in each vertical strip $|\Re z| \leq K$. We then have

$$\Delta_0(z) = dz^2 - cz + O(1), \quad |\Im z| \rightarrow \infty$$

uniformly in such a strip. Thanks to the assumption that $\Delta_0(i\eta) \neq 0$ for any $\eta \in \mathbb{R}$, Lemma 2.1 implies that there exists $\alpha > 0$ such that $\Delta_0^{-1}(z)$ is analytic in the strip $|\Re z| < \alpha$. In order to obtain (2.8), we distinguish between two cases.

The case that $d = 0$.

We write

$$\Delta_0^{-1}(z) = [-c(z+k)]^{-1} + R(z),$$

where $k > 2\alpha$. Clearly, in the strip $|\Re z| < \alpha$, $R(z)$ is analytic. Moreover, $R(z)$ satisfies $R(z) = O(|\Im z|)^{-2}$ uniformly as $|\Im z| \rightarrow \infty$. Consequently, if $\xi \geq 0$, then we can calculate the function G_0 by shifting the path of integration of integral in (2.9) as follows:

$$\begin{aligned} G(\xi) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi s} \Delta_0^{-1}(is) ds = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi s} [-c(is+k)^{-1} + R(is)] ds \\ &= -\frac{e^{-k\xi}}{2\pi c} + \frac{e^{-\alpha\xi}}{2\pi} \int_{\mathbb{R}} e^{i\xi s} R(\lambda_+^s - \epsilon + is) ds. \end{aligned}$$

The absolute convergence in last integral yields

$$|G_0(\xi)| \leq C e^{-\alpha\xi}, \quad \xi \geq 0$$

for some positive constant C . In the same manner, we can infer that

$$|G_0(\xi)| \leq C e^{-\alpha\xi}, \quad \xi \leq 0.$$

It is evident that the same reasoning works for $d > 0$ since $\Delta_0^{-1}(z) = O(|\Im z|^{-2})$ in the strip $|\Re z| \leq \alpha$. Thus (2.8) is completed. Furthermore, $(1 + |\eta|)\Delta_0^{-1}(i\eta) \in L^2$ implies that $G \in H^1$

and $\widehat{G'_0} = is\Delta_0^{-1}(is)$. In addition, it follows the same lines that

$$|G'_0(\xi)| \leq Ce^{-\alpha|\xi|}, \quad \xi \in \mathbb{R}.$$

We now solve the inhomogeneous problem

$$(2.11) \quad L_0 v = h, \quad h \in L^p$$

for $h \in L^p$. For given $h \in L^p$, let v be the convolution $v = G_+ * h$. Then by Young's inequality,

$$\|v\|_{L^p} \leq \|G_+\|_{L^1} \|h\|_{L^p}.$$

Also note that

$$\|v'\|_{L^p} \leq \|G'_+\|_{L^1} \|h\|_{L^p}$$

provided $d \neq 0$. To verify v is a solution to (2.11), it is sufficient to show that (2.11) holds everywhere for the function v , that is,

$$\int_{\mathbb{R}} \chi(\xi)(L_0 v)(\xi) d\xi = d \int_{\mathbb{R}} \chi'(\xi)v'(\xi) d\xi - c \int_{\mathbb{R}} \chi'(\xi)v(\xi) d\xi + \int_{\mathbb{R}} \chi(\xi)h(\xi) d\xi$$

for all C^∞ functions $\chi : \mathbb{R} \rightarrow \mathbb{C}$ of compact support. Indeed, it follows from the jump condition (2.10) and Fubini's theorem that

$$\begin{aligned} \int_{\mathbb{R}} \chi(\xi)(L_0 v)(\xi) d\xi &= a \int_{\mathbb{R}} \chi(\xi)(G_0 * h)(\xi) d\xi + b \int_{\mathbb{R}} \chi(\xi)(J * G_0 * h)(\xi) d\xi \\ &= \int_{\mathbb{R}} \chi(\xi)[b(J * G_0) + aG_0] * h(\xi) d\xi \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \chi(\xi)(cG'_0(\xi - \eta) - dG''_0(\xi - \eta)) d\xi \right] h(\eta) d\eta + \int_{\mathbb{R}} \chi(\eta)h(\eta) d\eta \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} (-c\chi'(\xi)G_0(\xi - \eta) + d\chi'(\xi)G'_0(\xi - \eta)) d\xi \right] h(\eta) d\eta + \int_{\mathbb{R}} \chi(\eta)h(\eta) d\eta \\ &= d \int_{\mathbb{R}} \chi'(\xi)v'(\xi) d\xi - c \int_{\mathbb{R}} \chi'(\xi)v(\xi) d\xi + \int_{\mathbb{R}} \chi(\xi)h(\xi) d\xi. \end{aligned}$$

Now, to complete the proof, we only need to show that $L_0 u = 0$ for some $u \in D(L_0)$ if and only if $u = 0$. In fact, by interpreting u as tempered distribution and taking the Fourier transform, we have

$$\Delta_0(i\eta)\widehat{u}(\eta) = 0.$$

Since $\Delta_0(i\eta) \neq 0$ for any $\eta \in \mathbb{R}$, \widehat{u} must be a zero distribution and hence $u = 0$. The proof is completed \square

We now construct the Green's function for a small perturbation L_q of L_0 . Namely $L_q = L_0 + Q$, here $Q : L^p \rightarrow L^p$ is a bounded linear operator defined by

$$(Qv)(\xi) = m(\xi)v(\xi) + n(\xi) \int_{\mathbb{R}} J(\xi - \eta)v(\eta) d\eta,$$

where $m, n \in L^\infty$.

Proposition 2.1. *Let L_0 be given by (2.5) and $L_q v(\xi) = (L_0 + Q)v(\xi)$. If*

$$\max\{\|m\|_{L^\infty}, \|n\|_{L^\infty}\} \leq \varepsilon$$

such that ε is sufficiently small, then $L_q : D(L_0) \rightarrow L^p$ is an isomorphism for $1 \leq p \leq \infty$. In addition, there exist positive constants ν, K , and the function $G_q : \mathbb{R}^2 \rightarrow \mathbb{C}$ satisfying the pointwise estimates

$$(2.12) \quad |G_q(\xi, \eta)| \leq K e^{\nu(\xi - \eta)}$$

such that

$$L_q^{-1} = \int_{\mathbb{R}} G_q(\xi, \eta) h(\eta) d\eta$$

for each $h \in L^p$.

Proof. The proof is similar to the proof of . We shall only sketch the proof of Proposition. Since $L_0 : D(L_0) \rightarrow L^p$ is an isomorphism, $L_q = (I - QL_0^{-1})L_0$. It then follows from Theorem 1.16 of [12] that

$$(2.13) \quad L_q^{-1} = L_0^{-1} \sum_{j=0}^{\infty} (QL_0^{-1})^j$$

as long as $\|QL_0^{-1}\| < 1$. Set

$$\Gamma_1(\xi, \eta) = m(\xi)G_0(\xi - \eta) + n(\xi) \int_{\mathbb{R}} J(\xi - s)G_0(s, \eta)ds.$$

In view of Lemma 2.2, $(QL_0^{-1})^j$ is an integral operator, whose kernel is defined inductively by

$$\Gamma_j(\xi, \eta) = \int_{\mathbb{R}} \Gamma_1(\xi, \tau) \Gamma_{j-1}(\tau, \eta) d\tau$$

for all $j \geq 2$. Thanks to (H1), a straightforward calculation shows that

$$(2.14) \quad |\Gamma_1(\xi, \eta)| \leq \varepsilon C e^{-\alpha|\xi - \eta|} + \varepsilon e^{-\alpha|\xi - \eta|} \int_{\mathbb{R}} J(s) e^{\alpha|s|} ds.$$

Therefore, there exists positive constant K_1 such that $|\Gamma_1(\xi, \eta)| \leq 2\varepsilon K e^{-\alpha|\xi - \eta|}$. In addition, by using (2.14), we infer that

$$|\Gamma_j(\xi, \eta)| \leq \Psi^{*j}(\xi - \eta), \quad \Psi(\xi) = 2\varepsilon K_1 e^{-\nu|\xi|},$$

where $\Psi^{*j} = \Psi * \Psi^{*(j-1)}$, is the j -fold convolution of Ψ with itself. By Lemma 5.1 of [15], we infer that

$$(2.15) \quad \sum_{j=1}^{\infty} |\Gamma_k(\xi, \eta)| \leq K_2 e^{-\nu|\xi - \eta|}.$$

Here $\nu = \sqrt{\alpha^2 - 4\varepsilon K \alpha}$ and $K_2 = \frac{2\varepsilon K \alpha}{\beta}$. Now Let

$$G_q(\xi, \eta) = G_0(\xi - \eta) + \int_{\mathbb{R}} G_0(\xi - s) \left(\sum_{j=1}^{\infty} \Gamma_j(s, \eta) \right) ds.$$

Then a direct calculation yields

$$|G_q(\xi, \eta)| \leq K e^{-\nu|\xi-\eta|}.$$

Furthermore, it is easy to see that

$$L_q^{-1}h = \int_{\mathbb{R}} G_q(\xi, \eta)h(\eta)d\eta.$$

Therefore, the proof is completed. \square

Next we consider the operator L defined by (2.4). Hereafter, we assume that

$$(2.16) \quad \lim_{\xi \rightarrow +\infty} a(\pm\xi) = a^{\pm}, \quad \lim_{\xi \rightarrow +\infty} b(\pm\xi) = b^{\pm},$$

where $a^+, a^-, b^+,$ and b^- are constants. Let $L_{\pm} : D(L_0) \rightarrow L^p$ be the operator defined by

$$L_{\pm}u = du'' - cu' + a^{\pm} + b^{\pm}J * u,$$

respectively.

Definition 2.2. *The operator L is called asymptotic hyperbolic if both L_+ and L_- are hyperbolic.*

Proposition 2.2. *Assume that 2.16 is satisfied and L is asymptotically hyperbolic. Suppose that there are bounded sequences $u_n \in D(L)$ and $h_n \in L^p$ such that $Lu_n = h_n$ and $h_n \rightarrow h^*$ in L^p . Then there exists a subsequence $u_{n'}$ and some $u^* \in D(L)$ such that $u_{n'} \rightarrow u^*$ in $D(L)$ and $Lu^* = h^*$. The same conclusion hold true for L^* .*

Proof. Due to the assumption, for any $\varepsilon > 0$, there exists $\tau(\varepsilon) > 0$ such that $|a(\pm\xi) - a^{\pm}| \leq \varepsilon$ and $|b(\pm\xi) - b^{\pm}| \leq \varepsilon$ whenever $\xi \geq \tau$. Now let

$$\theta_{\tau}(\xi) = \begin{cases} 1, & \xi \geq \tau, \\ 0, & \xi < \tau, \end{cases}$$

Let the bounded linear operators $Q^{\pm} : L^p \rightarrow L^p$, defined by

$$(Q^{\pm}v)(\xi) = \theta_{\tau}(\pm\xi)a(\xi) + \theta_{\tau}(\xi)b(\xi) \int_{\mathbb{R}} J(\xi - \eta)v(\eta)d\eta,$$

respectively. Set $L_{q^{\pm}} = L_{\pm} + Q^{\pm}$. It is clear that $L_{q^{\pm}}u_n = (L_{q^{\pm}} - L)u_n + f_n$. $p > 1$, then the embedding theorem implies that the sequence u_n is equicontinuous on any compact interval. In case of $d > 0$, u'_n is also equicontinuous on any compact interval. When $p = 1$, by means of an argument similar to one used in [15], it can be shown that the above conclusions are still true. Therefore there is a subsequence, still labeled by u_n , which converges to u^* uniformly on any compact interval for some function $u^* : \mathbb{R} \rightarrow \mathbb{C}$. Clearly, $u^* \in L^{\infty}$ since u_n is bounded in $W^{1,p}$.

Next we show that $u_n \rightarrow u^*$ in L^p if $1 \leq p < \infty$. By the assumption, there exists $\widehat{C} > 0$ such that $\|u_n\|_{L^{\infty}} \leq \widehat{C}$. Let

$$g_n(\xi) = K_2 \int_{\mathbb{R}} e^{-\mu|\xi-\eta|} |h_n(\eta)| d\eta, \quad g^*(\xi) = K_2 \int_{\mathbb{R}} e^{-\mu|\xi-\eta|} |h^*(\eta)| d\eta,$$

then it follows from (4.2) that

$$|u_n(\xi)| \leq \widehat{C}K_1e^{-\mu|\xi|} + g_n(\xi), \quad \xi \in \mathbb{R}.$$

By using the Hölder inequality and Young inequality, we find that $g_n, g^* \in L^p \cap L^\infty$, and

$$\lim_{n \rightarrow \infty} \|g_n - g^*\|_{L^\infty} = 0, \quad \lim_{n \rightarrow \infty} \|g_n - g^*\|_{L^p} = 0.$$

The generalized Lebesgue dominated convergence theorem implies $\|u_n\|_{L^p} \rightarrow \|u^*\|_{L^p}$. Finally, Brézis-Lieb Lemma yields

$$\lim_{n \rightarrow \infty} \|u_n - u^*\|_{L^p} = 0.$$

Due to (4.7), we have

$$u_n = \Lambda_\omega^{-1}((\lambda - \omega)Iu_n - Lu_n + h_n).$$

By passing the limit $n \rightarrow \infty$, we see that

$$u^* = \Lambda_\omega^{-1}((\lambda - \omega)Iu^* - Lu^* + h^*).$$

Namely,

$$(\Pi_L - \lambda I)u^* = h^*.$$

Furthermore, applying (4.3) or (4.4) to the difference $u_n - u^*$ yields

$$u_n \rightarrow u^* \text{ in } W^{1,p}(\text{or } W^{2,p} \text{ if } d > 0).$$

It remains to show that the assertion is valid when $p = \infty$. We first write for each u_n in the form

$$cu_n(\xi_1) = cu_n(\xi_2) + \int_{\xi_2}^{\xi_1} [L(\eta)u_n(\eta) - h_n(\eta)]d\eta, \quad \text{for } d = 0$$

and

$$du'_n(\xi_1) = du'_n(\xi_2) + \int_{\xi_2}^{\xi_1} [cu'_n(\eta) - Lu(\eta) + h_n(\eta)]d\eta, \quad \text{for } d > 0,$$

where $-\infty < \xi_2 < \xi_1 < \infty$. Since $(J * u_n)(\cdot)$ converges $(J * u^*)(\cdot)$ pointwise and $J * u_n$ is uniformly bounded, upon taking the limit, we find

$$cu^*(\xi_1) = cu^*(\xi_2) + \int_{\xi_2}^{\xi_1} [L(\eta)u^*(\eta) - h_n(\eta)]d\eta, \quad \text{for } d = 0$$

and

$$du^{*'}(\xi_1) = du^{*'}(\xi_2) + \int_{\xi_2}^{\xi_1} [cu^{*'}(\eta) - Lu(\eta) + h_n(\eta)]d\eta, \quad \text{for } d > 0.$$

Therefore, we have

$$d(u^*)''(\xi) - c(u^*)'(\xi) + (Lu)(\xi) = h(\xi), \quad \text{for any } \xi \in \mathbb{R}.$$

Applying (4.2) to $u_n - u^*$ yields that

$$|(u_n - u^*)(\xi)| \leq 2\widehat{C}K_1e^{-\mu|\xi|}d\xi + K_2\mu\|h_n - h^*\|_{L^\infty}.$$

Since $h_n \rightarrow h^*$ in L^∞ , for any $\varepsilon > 0$, there exist positive constants $N(\varepsilon)$ and $T(\varepsilon)$ such that $|(u_n - u^*)(\xi)| \leq \frac{1}{2}\varepsilon$ whenever $n > N(\varepsilon)$ and $|\xi| > T(\varepsilon)$. In addition, we already know that u_n uniformly converge u^* on any compact interval. Hence, there exists $\widetilde{N}(\varepsilon) > 0$ such that

$\|(u_n - u^*)\|_{L^\infty} \leq \varepsilon$ if $n > \tilde{N}(\varepsilon)$. Once again, the similar reasoning shows that $u_n \rightarrow u^*$ in $W^{1,\infty}$ (or $W^{2,\infty}$). Thus, the proof is completed. \square

Proposition 2.3. *Let (c, U) be the solution to (1.1), then there exist positive constants ν and C_ν such that*

$$(2.17) \quad |U'(\xi)| \leq C_\nu e^{(\lambda_+^s + \nu)\xi}, \quad \xi \geq 0$$

and

$$(2.18) \quad |U'(\xi)| \leq C_\nu e^{(\lambda_-^u - \nu)\xi}, \quad \xi \leq 0,$$

where $\nu < \min\{-\lambda_+^s, \lambda_-^u\}$.

Proof. We shall use Proposition 2.1 to derive (2.17) and (2.18). Recall that

$$M_\pm(\xi)v = [f_r(U, J * U) - f_r(\pm 1, \pm 1)]v + [f_s(U, J * U) - f_s(\pm 1, \pm 1)]J * v.$$

Due to (2.3), for any $\varepsilon > 0$, there exists $\tau > 0$ such that

$$\|M_+(\xi)\| \leq \varepsilon, \text{ as } \xi \geq \tau, \text{ and } \|M_-(\xi)\| \leq \varepsilon, \text{ as } \xi \leq -\tau.$$

Now let

$$(2.19) \quad \vartheta_+(\xi) = \begin{cases} 1, & \xi \geq \tau, \\ 0, & \xi < \tau, \end{cases} \quad \vartheta_-(\xi) = \begin{cases} 0, & \xi > -\tau, \\ 1, & \xi \leq -\tau. \end{cases}$$

We also set

$$\bar{L}_\pm(\xi) = L_\pm + \vartheta_\pm(\xi)M_\pm(\xi), \quad \bar{M}_\pm(\xi) = (1 - \vartheta_\pm(\xi))M_\pm(\xi).$$

Let V stand for U' and Let

$$(\Pi_{\bar{L}_\pm} v)(\xi) = dv''(\xi) - cv'(\xi) + \bar{L}_\pm v(\xi).$$

Then $(\Pi_{\bar{L}_\pm} V)(\xi) = -\bar{M}_\pm(\xi)V(\xi)$. Since $\bar{M}_\pm(\xi)$ is a bounded linear operator and ε can be made arbitrary small by manipulating τ , we can choose τ sufficiently large such that the operators $\Pi_{\bar{L}_\pm}$ satisfies the conditions of Proposition 2.1. Let $\bar{G}_+ : \mathbb{R}^2 \rightarrow \mathbb{C}$ denote the Green's function for $\Pi_{\bar{L}_+}$, which enjoys the estimate (2.12). Therefore, for every $\xi \in \mathbb{R}$,

$$\begin{aligned} V(\xi) &= \int_{-\infty}^{\infty} \bar{G}_+(\xi, \eta)[- \bar{M}_+(\eta)V(\eta)]d\eta \\ &= \int_{-\infty}^{\tau} \bar{G}_+(\xi, \eta)[-M_1(\eta)V(\eta)]d\eta, \end{aligned}$$

where we use the fact that $M_1(\eta) = 0$ for all $\eta \geq \tau$. Consequently, for any $\xi \geq \tau$, we have

$$\begin{aligned} |V(\xi)| &\leq C_1 \int_{-\infty}^{\tau} e^{(\lambda_+^s + \nu)(\xi - \eta)} \|M\|_{L^\infty} \|V\|_{L^\infty} d\eta \\ &\leq \tilde{C}_1 e^{(\lambda_+^s + \nu)\xi}. \end{aligned}$$

Since V is bounded on \mathbb{R} , it is possible to choose $C_\nu > 0$ such that the desired estimate (2.17) holds for all $\xi \geq 0$. Analogously,

$$|U'(\xi)| \leq C_\nu e^{(\lambda_-^u - \nu)\xi}, \quad \xi \leq 0.$$

The proof is completed. \square

Now we are ready to give the main result in this section

Theorem 2.1. *Let (c, U) be the solution to (1.1), then there exist positive constants D_1 and D_2 such that*

$$(2.20) \quad U(\xi) = 1 - D_1 e^{\lambda_+^s \xi} [1 + o(1)], \quad \text{as } \xi \rightarrow \infty,$$

and

$$(2.21) \quad U(\xi) = -1 + D_2 e^{\lambda_-^u \xi} [1 + o(1)], \quad \text{as } \xi \rightarrow -\infty,$$

where $\lambda_+^s < 0, \lambda_-^u > 0$ are the roots of $\Delta_{L\pm}(z)$, respectively.

Proof. We first show that

$$(2.22) \quad |M_+(\xi)V(\xi)| \leq C_2 e^{2(\lambda_+^s + \nu)\xi}, \quad \xi \geq 0,$$

$$(2.23) \quad |M_-(\xi)V(\xi)| \leq C_2 e^{2(\lambda_-^u - \nu)\xi}, \quad \xi \leq 0$$

hold true for some constant $C_2 > 0$. In fact, by mean value theorem, we have

$$|M_{\pm}(\xi)V(\xi)| \leq \tilde{K}[|U(\xi) \mp 1| + |J * (U \mp 1)(\xi)|][|V(\xi)| + |J * V(\xi)|]$$

for some positive constant \tilde{K} . It follows from proposition 2.3 that

$$|U(\xi) - 1| \leq (\lambda_+^s + \nu)^{-1} C_\nu e^{(\lambda_+^s + \nu)\xi}, \quad \xi \geq 0 \quad \text{and} \quad |U(\xi) + 1| \leq (\lambda_-^u - \nu)^{-1} C_\nu e^{(\lambda_-^u - \nu)\xi}, \quad \xi \leq 0.$$

Hence, for any $\xi \geq 0$,

$$\begin{aligned} & |J * (U - 1)(\xi)| \\ &= \left| \int_{\mathbb{R}} J(\eta)(U(\xi - \eta) - 1) d\eta \right| \leq \int_{-\infty}^{\xi} + \int_{\xi}^{\infty} J(\eta) |U(\xi - \eta) - 1| d\eta \\ &\leq \frac{C_\nu e^{(\lambda_+^s + \nu)\xi}}{(\lambda_+^s + \nu)} \int_{\mathbb{R}} J(\eta) e^{-(\lambda_+^s + \nu)\eta} d\eta + e^{(\lambda_+^s + \nu)\xi} \int_{\mathbb{R}} J(\eta) e^{-(\lambda_+^s + \nu)\eta} |U(\xi - \eta) - 1| d\eta \\ &\leq C' e^{(\lambda_+^s + \nu)\xi}. \end{aligned}$$

Similarly,

$$|J * (U + 1)(\xi)| \leq C e^{(\lambda_-^u - \nu)\xi}, \quad \xi \leq 0,$$

and

$$|J * V(\xi)| \leq \begin{cases} C_3 e^{(\lambda_+^s + \nu)\xi}, & \text{if } \xi \geq 0, \\ C_3 e^{(\lambda_-^u - \nu)\xi}, & \text{if } \xi \leq 0, \end{cases}$$

Therefore, (2.22) follows.

Now set $h_{\pm}(\xi) = -M_{\pm}(\xi)U(\xi)$. As long as ν is sufficiently small, there exists $\iota > 0$ such that $2(\lambda_+^s + \nu) \leq \lambda_+^s - \iota$ and $2(\lambda_-^u - \nu) \geq \lambda_-^u + \iota$. Therefore, we have

$$(2.24) \quad |h_+(\xi)| \leq C_2 e^{(\lambda_+^s - \iota)\xi}, \quad \text{for any } \xi \geq 0, \quad |h_-(\xi)| \leq C_2 e^{(\lambda_-^u + \iota)\xi}, \quad \text{for any } \xi \leq 0.$$

In addition, due to the boundedness of $U \mp 1$ and $J * (U \mp 1)$, it is easy to see that

$$(2.25) \quad |h_+(\xi)| = O(e^{(\lambda_+^s - \nu)\xi}), \quad \text{as } \xi \rightarrow -\infty, \quad |h_-(\xi)| = O(e^{(\lambda_+^s + \nu)\xi}), \quad \text{as } \xi \rightarrow \infty.$$

Clearly, we have

$$(2.26) \quad dV'' - cV' + L_\pm V = h_\pm(\xi).$$

In particular, when $d = 0$,

$$(2.27) \quad |V'(\xi)| \leq |b^\pm| |J * V(\xi)| + |a^\pm| |V(\xi)| + |h_\pm(\xi)|.$$

We also observe that h_\pm is differentiable and

$$|h'_\pm(\xi)| \leq K' [|V(\xi)|^2 + |V(\xi)| |J * V(\xi)| + |U(\xi) \mp 1| |V'(\xi)| + |U(\xi) \mp 1| |J * V'(\xi)|].$$

Therefore, it follows from (2.27) that

$$(2.28) \quad |h'_+(\xi)| \leq C_4 e^{(\lambda_+^s - \nu)\xi}, \quad \text{for any } \xi \geq 0, \quad |h'_-(\xi)| \leq C_4 e^{(\lambda_+^s + \nu)\xi}, \quad \text{for any } \xi \leq 0.$$

Next, we show (2.20). Thanks to (2.24) and (2.25), $\widehat{h}_+(z)$ is analytic in the strip $0 \leq \text{Im}z \leq 2\epsilon - \lambda_+^s$, where $0 < 2\epsilon < \iota$ and $\widehat{g}(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-izs} g(s) ds$. In case $d = 0$, let $h_+^\rho(\xi) = h_+(\xi) e^{\rho\xi}$, where $\rho \in (0, 2\epsilon - \lambda_+^s)$. Due to (2.28), for each $\rho \in (0, 2\epsilon - \lambda_+^s)$, $h_+^\rho \in W^{1,p}(\mathbb{R})$ for any $p \geq 1$. Furthermore, we have

$$|\eta| |\widehat{h}_+(\eta + i\rho)| = |\eta| |\widehat{h}_+^\rho(\eta)| = |\eta \widehat{h}_+^\rho(\eta)| = |\widehat{\partial_\eta h_+^\rho}(\eta)| \leq \|\partial_\eta h_+^\rho\|_{L^1}, \quad \eta \in \mathbb{R}.$$

Therefore, in the strip $0 \leq \text{Im}z \leq 2\epsilon - \lambda_+^s$,

$$|\widehat{h}_+(z)| = O(|\text{Re}z|^{-1}), \quad |\text{Re}z| \rightarrow \infty.$$

In the strip $|\text{Re}z| \leq D$ with any fixed $D > 0$, we have that $\Delta_{L_+}(z) = O(|\text{Im}z|)$ ($= O(|\text{Im}z|^2)$ if $d > 0$) uniformly as $|\text{Im}z| \rightarrow \infty$. Consequently, $\widehat{h}_+(-iz) \Delta_{L_+}^{-1}(z) = O(|\text{Im}z|)^{-2}$ for any $z \in \mathbb{C}$ with $0 < \text{Re}z \leq \lambda_+^s - 2\epsilon$ and $\Delta_{L_+}(z) \neq 0$.

Since Π_{L_+} is an isomorphism, V is the unique solution to $dv'' - cv' + L_+v = h_+$. By using Fourier transform and shifting the integrating path, when $\xi \geq 0$, we find

$$\begin{aligned} V &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\eta\xi} \widehat{h}_+(\eta)}{\Delta_+(i\eta)} d\eta = \frac{-i}{2\pi} \int_{\mathbb{R}} \frac{e^{i\eta\xi} \widehat{h}_+(-i(i\eta))}{\Delta_+(i\eta)} d(i\eta) \\ &= \sum \text{Res} \frac{e^{z\xi} \widehat{h}_+(-iz)}{\Delta_+(z)} \Big|_{\lambda_+^s - \epsilon \leq \text{Re}z \leq 0} + \frac{-i}{2\pi} \int_{\text{Re}z = \lambda_+^s - \epsilon} \frac{e^{z\xi} \widehat{h}_+(-iz)}{\Delta_{L_+}(z)} dz. \end{aligned}$$

Here we choose ϵ such that $\Delta_{L_+}(\lambda_+^s - \epsilon + i\eta) \neq 0$ for any $\eta \in \mathbb{R}$. Clearly, The last integral absolutely converges.

Let $\Upsilon_{\lambda_+^s - \epsilon} = \{z \in \mathbb{C} | \Delta_{L_+}(z) = 0, \lambda_+^s - \epsilon < \text{Re}z < 0\}$. Since $\widehat{h}(z)$ is analytic in the strip $0 \leq \text{Im}z \leq 2\epsilon - \lambda_+^s$, in the strip $\lambda_+^s - 2\epsilon \leq \text{Re}z \leq 0$, $\widehat{h}(-iz) \Delta_{L_+}^{-1}(z)$ is meromorphic and only has poles which may occur at $z \in \Upsilon_{\lambda_+^s - \epsilon}$. We claim that $\widehat{h}(iz) \Delta_{L_+}^{-1}(z)$ has a simple pole at $z = \lambda_+^s$. Suppose this is not true. In virtue of Lemma 2.1, in the strip $\lambda_+^s - \epsilon < \text{Re}z < 0$, either all the poles of $\widehat{h}(-iz) \Delta_{L_+}^{-1}(z)$ occur at $z \in \Upsilon_{\lambda_+^s - \epsilon}$ with $\text{Re}z < \lambda_+^s$ or $\widehat{h}(-iz) \Delta_{L_+}^{-1}(z)$ is analytic. For the latter, $V(\xi) = O(e^{(\lambda_+^s - \epsilon)\xi})$, as $\xi \rightarrow \infty$. Certainly, $h_+(\xi) = O(e^{2(\lambda_+^s - \epsilon)\xi})$ and

$\widehat{h}(iz)$ is analytic in the strip $2\lambda_+^s - \varkappa$ for some $0 < \varkappa \leq 2\epsilon$. Hence the path of integration can be shifted to the line $\operatorname{Re} z = 2\lambda_+^s - \varkappa$. Consequently, one of the following cases must occur.

Case I The set Z is not empty, where

$$Z = \{\operatorname{Re} z \in \mathbb{R}^- \mid \Delta_{L_+}(z) = 0, \widehat{h}(-iz) \text{ is analytic at } z \text{ and } \widehat{h}(-iz)\Delta_{L_+}^{-1}(z) \text{ has poles at } z\}.$$

Case II $V(\xi) = O(e^{-b\xi})$ for any $b \in \mathbb{R}^+$.

Next we show that both case I and II are impossible. We start with case (I). Let $\varrho = \sup Z$. Recall that λ_+^s is the only real zero of Δ_{L_+} in the half plane $\operatorname{Re} z \leq 0$. By lemma 2.1, we may assume $\varrho \pm \mu_m i$ with $\mu_m > 0 (1 \leq m < \infty)$ are all the element of Z with real part equal to ϱ . Suppose that $\widehat{h}(-iz)\Delta_{L_+}^{-1}(z)$ has a pole of order $l_m + 1$ at $\varrho + \mu_m i$. Then

$$\begin{aligned} V(\xi) &= \sum \operatorname{Res}(e^{\xi z} \widehat{h}(iz) \Delta_{L_+}^{-1}(z))_{\operatorname{Re} z = \varrho} + o(e^{\varrho \xi}) \\ &= \sum_m p_{l_m}(\xi) e^{\varrho \xi} \cos(\mu_m \xi + k_m) + o(e^{\varrho \xi}), \end{aligned}$$

where p_{l_m} are real polynomials and $k_m \in \mathbb{R}$. Thus, $V(\xi) = \xi^N e^{\varrho \xi} (q(\xi) + O(\xi^{-1}))$ as $\xi \rightarrow \infty$ for some $N > 0$, where q is a quasiperiodic function of mean value zero. According to [16],

$$\liminf_{\xi \rightarrow \infty} q(\xi) < 0 < \limsup_{\xi \rightarrow \infty} q(\xi).$$

Consequently, $V(\xi_1) < 0$ for some $\xi_1 > 0$. This contradicts the fact that $V(\xi) > 0$ for any $\xi \in (-\infty, \infty)$. Therefore case(I) never occurs.

For the case II, we define

$$V_b = \int_{\mathbb{R}} V(\xi) e^{b\xi}, \quad b \in \mathbb{R}^+.$$

Note that

$$\int_{\mathbb{R}} e^{b\xi} J * V(\xi) d\xi = \int_{\mathbb{R}} e^{b\eta} V(\eta) \int_{\mathbb{R}} e^{b(\xi-\eta)} J(\xi-\eta) d\xi d\eta = V_b \int_{\mathbb{R}} e^{b\xi} J(\xi) d\xi.$$

Let

$$\underline{a} = \min_{[-1,1] \times [-1,1]} f_r(r, s), \quad \underline{b} = \min_{[-1,1] \times [-1,1]} f_s(r, s).$$

Due to (H2) and (H3), $\underline{a} > -\infty$ and $\underline{b} > 0$. Consequently,

$$(2.29) \quad cV' - dV'' \geq \underline{a}V + \underline{b}J * V$$

Multiplying each side of (2.29) by $e^{b\xi}$ and integrating by part yield

$$(-cb - db^2 - \int_{\mathbb{R}} J(\xi) e^{b\xi} d\xi) V_b \geq \underline{a} V_b,$$

thus

$$-cb - db^2 - \underline{b} \int_{\mathbb{R}} J(\xi) e^{b\xi} d\xi \geq \underline{a}.$$

Since $-cb - db^2 - \int_{\mathbb{R}} J(\xi) e^{b\xi} d\xi \rightarrow -\infty$, as $b \rightarrow \infty$, we arrive at a contradiction. This implies that Case (II) can not occur. Therefore, $e^{z\xi} \widehat{h}(-iz) \Delta_{L+}^{-1}(z)$ has a simple pole at $z = \lambda_+^s$, and

$$\begin{aligned} V(\xi) &= \frac{e^{\lambda_+^s \xi} \widehat{h}_+(-i\lambda_+^s)}{\Delta'_{L+}(\lambda_+^s)} + \frac{1}{2\pi} \int_{\operatorname{Re} z = \lambda_+^s - \epsilon} \frac{e^{z\xi} \widehat{h}_+(-iz)}{\Delta_{L+}(z)} dz \\ &= \frac{e^{\lambda_+^s \xi} \int_{\mathbb{R}} h(\eta) e^{-\lambda_+^s \eta} d\eta}{\int_{\mathbb{R}} \eta J(\eta) e^{-\lambda_+^s \eta} d\eta - c} + \frac{e^{(\lambda_+^s - \epsilon)\xi}}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi s} \widehat{h}_+(\eta + i(\epsilon - \lambda_+^s))}{\Delta_{L+}(\lambda_+^s - \epsilon + i\eta)} d\eta. \end{aligned}$$

Now let

$$\gamma^+ = \frac{\int_{\mathbb{R}} h(\eta) e^{-\lambda_+^s \eta} d\eta}{\int_{\mathbb{R}} \eta J(\eta) e^{-\lambda_+^s \eta} d\eta - c}, \quad V^+(\xi) = \frac{e^{(\lambda_+^s - \epsilon)\xi}}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi s} \widehat{h}_+(\eta + i(\epsilon - \lambda_+^s))}{\Delta_{L+}(\lambda_+^s - \epsilon + i\eta)} d\eta.$$

Clearly, $V^+(\xi) = o(e^{\lambda_+^s \xi})$, as $\xi \rightarrow +\infty$. The positivity of V forces that $\gamma^+ > 0$. Thus,

$$U'(\xi) = \gamma^+ e^{\lambda_+^s \xi} + o(e^{\lambda_+^s \xi}), \quad \xi \rightarrow +\infty.$$

By considering the equation $cV' - dV'' = L_- V + h_-$ and arguing analogously, we may find

$$U'(\xi) = \gamma^- e^{\lambda_-^u \xi} + o(e^{\lambda_-^u \xi}), \quad \xi \rightarrow -\infty$$

for some constant $\gamma^- > 0$. With the boundary conditions $U(\pm\infty) = \pm 1$, we are readily to obtain the desired conclusions. \square

The uniqueness of monotone traveling wave U for nonlocal Allen-Cahn equation (1.1) with $d = 0$ were established in [4] and later for the general equation (1.1) in [8]. In those works, the uniqueness of speed and profile of traveling wave solution U are obtained by means of a comparison principle and sub- and super solution techniques. Here we provided a technically different and simplified proof for the uniqueness of U .

Corollary 2.1. *Assume that (H1)-(H5) are satisfied. Then there exists a unique $c^* \in \mathbb{R}$ such that equation (1.1) possesses a solution satisfying (2.1) if and only if $c = c^*$ and the traveling wave solution U is unique up to translation of ξ .*

Proof. We shall argue by contradiction. Suppose that there exist (c_i, U_i) satisfying (1.1) with $c_1 < c_2$, $i = 1, 2$. We may assume that one of these solutions has speed c^* . By Theorem 2.1, both solutions satisfy

$$U_i(\xi) = \begin{cases} -1 + n_i e^{\varrho_i^u \xi} + o(e^{\varrho_i^u \xi}), & \xi \rightarrow -\infty, \\ 1 - \tilde{n}_i e^{\varrho_i^s \xi} + o(e^{\varrho_i^s \xi}), & \xi \rightarrow \infty \end{cases}$$

for some $n_i, \tilde{n}_i > 0$ and $\varrho_i^u > 0, \varrho_i^s < 0$. Furthermore, ϱ_i^u and ϱ_i^s satisfy

$$\begin{aligned} c_i \varrho_i^s - d(\varrho_i^s)^2 - a^+ - b^+ \int_{\mathbb{R}} J(s) e^{-\varrho_i^s s} ds &= 0, \\ c_i \varrho_i^u - d(\varrho_i^u)^2 - a^- - b^- \int_{\mathbb{R}} J(s) e^{-\varrho_i^u s} ds &= 0. \end{aligned}$$

In view of the proof of lemma 2.1, it is easy to see that

$$\varrho_1^s < \varrho_2^s < 0, \quad 0 < \varrho_1^u < \varrho_2^u.$$

Thus, $U_2(\xi) < U_1(\xi)$ for all sufficiently large $|\xi|$. This together with the monotonicity of U_i justify that we can choose $\tau \in \mathbb{R}$ and replace $U_2(\xi)$ by $U_2(\xi + \tau)$ such that $U_2(\xi) \leq U_1(\xi)$ for all $\xi \in \mathbb{R}$ and $U_2(\xi^*) = U_1(\xi_0)$ for some ξ_0 . Consequently, $U_2'(\xi_0) = U_1'(\xi_0)$ and $U_2''(\xi_0) \leq U_1''(\xi_0)$. Moreover, (H2) and the fact that $J * U_1(\xi_0) \geq J * U_2(\xi_0)$ imply that $f(U_1(\xi_0), J * U_1(\xi_0)) \geq f(U_2(\xi_0), J * U_2(\xi_0))$. By plugging these relations into (1.1), we find

$$0 = dU_1''(\xi_0) - c_1 U_1'(\xi_0) + f(U_1(\xi^*), J * U_1(\xi_0)) > dU_2''(\xi_0) - c_2 U_2'(\xi_0) + f(U_2(\xi_0), J * U_2(\xi_0)) = 0.$$

The contradiction completes the proof. \square

3. SPECTRAL ANALYSIS OF TRAVELING WAVE U

In this section, we study the spectrum of the operator Π_L . Recall that

$$(3.1) \quad (\Pi_L u)(\xi) := du''(\xi) - cu'(\xi) + f_r(U, J * U)u(\xi) + f_s(U, J * U)(J * u)(\xi)$$

and

$$(3.2) \quad (\Pi_{L\pm} u)(\xi) := du''(\xi) - cu'(\xi) + a^\pm u(\xi) + b^\pm(J * u)(\xi).$$

Clearly, the equation $(\Pi_L - \lambda I)u = 0$ is equivalent to

$$(3.3) \quad du''(\xi) - cu'(\xi) + L(\xi)u(\xi) - \lambda u(\xi) = 0.$$

Let

$$L^*(\xi)v(\xi) = f_r(U(\xi), J * U(\xi))v(\xi) + \int_{\mathbb{R}} J(\xi - \eta)f_s(U(\eta), J * U(\eta))v(\eta)d\eta.$$

The adjoint equation of (3.3) is the equation

$$(3.4) \quad dv''(\xi) + cv'(\xi) + L(\xi)v(\xi) - \bar{\lambda}v(\xi) = 0,$$

where $\bar{\lambda}$ denotes the conjugate of λ . We define the formally adjoint operator $(\Pi_L - \lambda I)^*$ of $(\Pi_L - \lambda I)$ to be

$$((\Pi_L - \lambda I)^* v)(\xi) = dv''(\xi) + cv'(\xi) + L^*(\xi)v(\xi) - \bar{\lambda}v(\xi).$$

It is easy to see that

$$(3.5) \quad \int_{\mathbb{R}} \overline{v(\xi)}((\Pi_L - \lambda I)u)(\xi)d\xi = \int_{\mathbb{R}} \overline{((\Pi_L - \lambda I)^* v)(\xi)}u(\xi)d\xi,$$

and

$$(3.6) \quad (\Pi_L - \lambda I)^* = \Pi_L^* - \bar{\lambda}I,$$

where $u \in W^{1,p}$, $v \in W^{1,q}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Similarly, the formally adjoint operators $(\Pi_{L\pm} - \lambda I)^*$ of $\Pi_{L\pm} - \lambda I$ are defined by

$$(3.7) \quad ((\Pi_{L\pm} - \lambda I)^* v)(\xi) = dv''(\xi) + cv'(\xi) + L_\pm v(\xi) - \bar{\lambda}v(\xi).$$

Throughout the rest of the paper, we let $X := L^p(\mathbb{R}, \mathbb{C})$, $1 \leq p \leq \infty$. $\Re X$ is considered as an ordered Banach space with a positive cone X_+ , where $\Re X = \{\operatorname{Re} u | u \in X\}$ and $X_+ = \{w \in \Re X | w \geq 0\}$. It is well known that X_+ is generating, normal, (see [1] for more

details). For $\varphi \in \Re X$, we write $\varphi \gtrsim 0$ if $\varphi \in X_+$ and $\varphi \neq 0$, $\varphi \gg 0$ if $\varphi(\xi) > 0$ for each $\xi \in \mathbb{R}$. An operator $A : X \rightarrow X$ is called positive if $AX_+ \subseteq X_+$.

Definition 3.1. An operator A is said to be resolvent positive if the resolvent set $\rho(A)$ of A contains an interval (α, ∞) and $(\lambda I - A)^{-1}$ is positive for sufficiently large $\lambda \in \rho(A) \cap \mathbb{R}$.

In sequel, we follow [13] to define the normal points and the essential spectrum of an operator A on a Banach space. Namely, a normal point of A is a complex number in the resolvent set $\rho(A)$ or an isolated eigenvalue of A with finite multiplicity. The complement of the set of normal points is called the essential spectrum of A denoted by $\sigma_{ess}(A)$. We denote the spectral bound of an operator A by

$$\mathfrak{s}(A) = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\}.$$

We also let $\bar{\tau} = \max\{a^+ + b^+, a^- + b^-\}$ and $\underline{\tau} = \min\{a^+ - b^+, a^- - b^-\}$, where $a^\pm = f_r(\pm 1, \pm 1)$, $b^\pm = f_s(\pm 1, \pm 1)$.

Theorem 3.1. Consider the linear operator $\Pi_L : L^p \rightarrow L^p$ defined by (3.1), which corresponds to the variational equation of (1.1) at U , that is

$$(\Pi_L u)(\xi) = du''(\xi) - cu'(\xi) + f_r(U, J * U)u(\xi) + f_s(U, J * U)(J * u)(\xi)$$

and its formally adjoint operator $\Pi_L^* : L^q \rightarrow L^q$ defined by

$$(\Pi_L^* u)(\xi) = du''(\xi) + cu'(\xi) + f_r(U, J * U)u(\xi) + (J * f_s(U, J * U)u)(\xi),$$

where $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. $q = \infty$ if $p = 1$, and $q = 1$ if $p = \infty$. Then

Case $d = 0$.

(i) Let $\Omega_+ = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda > \bar{\tau}\}$ and $\Omega_- = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda < \underline{\tau}\}$. λ is a isolated eigenvalue with finite algebraic multiplicity if $\lambda \in (\Omega_+ \cup \Omega_-) \cap \sigma(\Pi_L)$. Furthermore, suppose ψ is an eigenfunction corresponding to λ , then

$$(3.8) \quad |\psi(\xi)| \leq C_\lambda e^{-\mu|\xi|}, \quad \xi \in \mathbb{R}$$

for some positive constants C_λ and μ .

(ii) $\sigma_{ess}(\Pi_L) \subseteq \{\lambda \in \mathbb{C} | \underline{\tau} \leq \operatorname{Re} \lambda \leq \bar{\tau}\}$.

(iii) $\mathfrak{s}(\Pi_L) = \mathfrak{s}(\Pi_L^*) = 0$ and 0 is a simple eigenvalue for both Π_L and Π_L^* .

(iv) $\dim \mathcal{N}(\Pi_L) = \dim \mathcal{N}(\Pi_L^*) = \operatorname{codim} \mathcal{R}(\Pi_L) = \operatorname{codim} \mathcal{R}(\Pi_L^*) = 1$. Moreover, Π_L^* has a positive eigenfunction Ψ corresponding to the eigenvalue 0, and

$$(3.9) \quad \mathcal{R}(\Pi_L) = \{h \in L^p | \int_{\mathbb{R}} \Psi(\xi) h(\xi) d\xi = 0\}.$$

(v) There exist $\varpi > 0$ such that the set $\{\lambda \in \mathbb{C} | \operatorname{Re} \lambda < -\varpi, \text{ or } \operatorname{Re} \lambda \geq 0 \text{ and } \lambda \neq 0\} \subset \rho(\Pi_L)$, where $\rho(\Pi_L)$ denotes the resolvent set of Π_L .

Case $d > 0$.

(i) Let $\Xi = \{\lambda \in \mathbb{C} | |\operatorname{Im}\lambda| > \sqrt{\bar{\tau} - \operatorname{Re}\lambda} + (b^- \wedge b^+), \operatorname{Re}\lambda \leq \bar{\tau}\} \cup \{\operatorname{Re}\lambda > \bar{\tau}\}$. Suppose that $\lambda \in \Xi \cap \sigma(\Pi_L)$, then λ is a isolated eigenvalue with finite algebraic multiplicity. Furthermore, if ψ is an eigenfunction corresponding to λ then

$$|\psi(\xi)| \leq C_\lambda e^{-\mu|\xi|}, \quad \xi \in \mathbb{R},$$

where C_λ and μ are positive constants.

(ii) $\sigma_{\text{ess}}(\Pi_L) \subseteq \mathbb{C} \setminus \Xi$.

The assertions (iii) and (iv) stated above remain true.

(v) the set $\{\lambda \in \mathbb{C} | \operatorname{Re}\lambda \geq 0 \text{ and } \lambda \neq 0\} \subset \rho(\Pi_L)$,

Proof. We shall first prove that Π_L is resolvent positive. The proof for Π_L^* is same. Let $\tilde{\lambda} > 0$ be sufficiently large and write $\tilde{\lambda} = \lambda^* - \underline{\lambda}$ such that $\max_{0 \leq U \leq 1} |f_r(U, J * U)| \leq \underline{\lambda} < \infty$. Then we have $(\tilde{\lambda}I - \Pi_L) = (\lambda^*I + c\partial - d\partial^2) - (\underline{\lambda}I + L)$, where $(Lv)(\xi) := f_r(U, J * U)v(\xi) + f_s(U, J * U)(J * v)$ and ∂ denotes differentiation. According to [17](see section 1.6), As long as λ^* is sufficiently large, $(\lambda^*I + c\partial - d\partial^2)$ is invertible and positive. In particular, $\|(\lambda^*I + c\partial - d\partial^2)^{-1}\| \leq |\frac{k}{\lambda^*}|$ for some $k > 0$ provided $d > 0$, and $\|(\lambda^*I + c\partial)^{-1}\| \leq |\frac{c}{\lambda^*}|$. Since L is a bounded operator, $\tilde{\lambda}I - \Pi_L$ is invertible provided $\tilde{\lambda}$ is sufficiently large. Therefore, we have

$$\begin{aligned} (\tilde{\lambda}I - \Pi_L)^{-1} &= [(\lambda^*I + c\partial - d\partial^2) - (\underline{\lambda}I + L)]^{-1} \\ &= (\lambda^*I + c\partial - d\partial^2)^{-1} [I - (\underline{\lambda}I + L)(\lambda^*I + c\partial - d\partial^2)^{-1}] \\ &= (\lambda^*I + c\partial - d\partial^2)^{-1} \sum_{j=0}^{\infty} [(\underline{\lambda}I + L)(\lambda^*I + c\partial - d\partial^2)^{-1}]^j. \end{aligned}$$

Note that $(\underline{\lambda}I + L)$ is positive. So the above Neumann series is a sum of positive operators and hence is positive. Clearly, for any $\lambda > \tilde{\lambda}$, $(\lambda I - \Pi_L)$ is invertible and positive. Hence Π_L is resolvent positive.

Next, we prove the statements (i)-(v) for the case that $d = 0$. The proof for case that $d > 0$ follows the same lines and shall be omitted.

By [17] again, there exists $\varpi > 0$ sufficient large such that $(\lambda I + c\partial)$ is invertible and $\|(\lambda I + c\partial)^{-1}\| \leq |\frac{c}{\lambda}|$ whenever $\lambda \leq -\varpi$. With the same reasoning used previously, we see that $\lambda I - \Pi_L$ is invertible provided $\lambda \leq -\varpi$. This prove the part of (v).

Notice that both Ω_+ and Ω_- are connected open subsets of \mathbb{C} . Due to Proposition 4.3 and Lemma 4.2 in Appendix, both $(\Pi_L - \lambda I)$ and $(\Pi_L^* - \lambda I)$ are semi-Fredholm operators whenever $\lambda \in \Omega_- \cup \Omega_+$. Since $\rho(\Pi_L) \cap \Omega_+ \neq \emptyset$ and $\rho(\Pi_L) \cap \Omega_- \neq \emptyset$, according the first paragraph on p243 of [12], the followings hold true:

(a1) $(\Pi_L - \lambda I)$ is Fredholm of index zero if $\lambda \in \Omega_- \cup \Omega_+$.

(a2) Suppose that $\lambda \in \sigma(\Pi_L) \cap (\Omega_+ \cup \Omega_-)$, then λ is a isolated eigenvalue with finite algebraic multiplicity.

Furthermore, Lemma 4.1 implies (3.8). Therefore (i) is completed. As a consequence of (i), (ii) is true. Next, we show (iii) and (iv). Analogously, (a1) and (a2) remain valid for Π_L^* . Notice that $0 \in \sigma(\Pi_L)$, and so $\mathfrak{s}(\Pi_L) \geq 0 > -\infty$. By [20], the resolvent positivity yields that $\mathfrak{s}(\Pi_L) \in \sigma(\Pi_L)$. In particular, $\mathfrak{s}(\Pi_L) \in \sigma(\Pi_L) \cap \Omega_+$ since $\bar{\tau} < 0$. Therefore, (a1) and (a2) imply that $\text{Ind}(\Pi_L - \mathfrak{s}(\Pi_L)I) = 0$ and $\mathfrak{s}(\Pi_L)$ is a isolated eigenvalue with finite algebraic multiplicity. It follows from Lemma 4.2 that

$$\text{codim}\mathcal{R}(\Pi_L^* - \mathfrak{s}(\Pi_L)I) = \text{codim}\mathcal{R}(\Pi_L - \mathfrak{s}(\Pi_L)I)^* \geq \dim\mathcal{N}(\Pi_L - \mathfrak{s}(\Pi_L)I) > 0.$$

Consequently, $\mathfrak{s}(\Pi_L) \in \sigma(\Pi_L^*)$. By the resolvent positivity of Π_L^* , we infer that $\mathfrak{s}(\Pi_L^*) \in \sigma(\Pi_L^*)$ and $\mathfrak{s}(\Pi_L^*) \geq \mathfrak{s}(\Pi_L) \geq 0$. Moreover, $\mathfrak{s}(\Pi_L^*)$ has a positive eigenfunction Ψ . Suppose that $\mathfrak{s}(\Pi_L^*) > 0$. Observe that $U'(\xi)$ is an eigenvalue of Π_L corresponding to eigenvalue 0 and hence $\mathfrak{s}(\Pi_L^*)U'(\xi) \in \mathcal{R}(\Pi_L - \mathfrak{s}(\Pi_L^*)I)$. Since $(\Pi_L - \mathfrak{s}(\Pi_L^*)I)^* = \Pi_L^* - \mathfrak{s}(\Pi_L^*)I$, it follows from lemma 4.2 in Appendix that

$$\mathfrak{s}(\Pi_L^*) \int_{\mathbb{R}} \Psi(\xi)U'(\xi)d\xi = 0,$$

which is impossible since $U' \gg 0$. Thus $\mathfrak{s}(\Pi_L^*) = \mathfrak{s}(\Pi_L) = 0$. We now prove the simplicity of eigenvalue 0, without loss of generality, we assume that $c > 0$. We first show that $\mathcal{N}(\Pi_L) = \text{span}\{U'\}$. Suppose this not true, then there is an eigenfunction ψ associated with eigenvalue 0 such that $\psi \neq tU'$ for all $t \in \mathbb{R}$. Obviously, $\psi \in W^{2,p}(\mathbb{R})$. In view of Theorem 2.1, $|\psi(\xi)| = O(e^{\lambda^{\frac{a}{2}}\xi})$, as $\xi \rightarrow \infty$, and $|\psi(\xi)| = O(e^{\lambda^{\frac{a}{2}}\xi})$, as $\xi \rightarrow -\infty$. Due to the positivity of U' , there exist t such that $tU' + \psi \geq 0$. Let $\bar{t} = \inf\{t \in \mathbb{R} : tU' + \psi \geq 0\}$. Obviously, $\bar{t}U' + \psi \neq 0$. Set $\bar{w} = \bar{t}U' + \psi$ and $\Sigma = \{\xi \in \mathbb{R} | \bar{w}(\xi) = 0\}$. Note that Σ is not empty by our assumption. Furthermore, Σ is a close set and $\Sigma \setminus \text{inter}\Sigma \neq \emptyset$. Let $\xi_0 \in \Sigma \setminus \text{inter}\Sigma$. Certainly, for each $\varepsilon > 0$, there is a point $\xi_\varepsilon \in (\xi_0 - \frac{1}{2}\varepsilon, \xi_0 + \frac{1}{2}\varepsilon)$ such that $\bar{w}(\xi_\varepsilon) > 0$. Since, for any $\gamma > \max_{0 \leq U \leq 1} |f_r(U, J * U)|$, $c\bar{w}' + \gamma\bar{w} = (L\bar{w} + \gamma\bar{w}) \gneq 0$, simple calculation shows that

$$\bar{w}(\xi) = \int_{-\infty}^{\xi} e^{-\frac{\gamma}{c}(\xi-\eta)} (L + \gamma I)\bar{w}(\eta)d\eta, \quad \xi \in \mathbb{R}.$$

Clearly, $\bar{w}(\xi) > 0$ for any $\xi \geq \xi_0 + \varepsilon$. Thanks to (H1), there exist a, b with $b > a > 0$ such that $(-b, -a) \cup (a, b) \subseteq \text{supp}J$. Since ε can be chosen sufficiently small such that $\varepsilon < a$, we find that $\text{supp}J(\xi_0 - \cdot) \cap \text{supp}\bar{w}(\cdot)$ contains a nonempty open interval. Hence $J * \bar{w}(\xi_0) > 0$. On the other hand, $\bar{w}'(\xi_0) = 0$ implies that

$$0 \leq f_s(U, J * U)(J * \bar{w})(\xi_0) \leq (\Pi_L \bar{w})(\xi_0) = 0.$$

(H3) forces that $J * \bar{w}(\xi_0) = 0$, thus we reach a contradiction. The contradiction leads to the desired conclusion that $\mathcal{N}(\Pi_L) = \text{span}\{U'\}$. As mentioned early, we can similarly show that $\mathcal{N}(\Pi_L) = \text{span}\{U'\}$ for the case that $d > 0$. However, the proof is much simpler. Indeed, we have $c\bar{w}' - d\bar{w}'' + \gamma\bar{w} = (L\bar{w} + \gamma\bar{w}) \gneq 0$, where γ is the constant same as one defined above.

Then

$$\overline{w}(\xi) = \int_{-\infty}^{\xi} e^{\mu_-} (L + \gamma I) \overline{w}(\eta) d\eta + \int_{\xi}^{\infty} e^{\mu_+} (L + \gamma I) \overline{w}(\eta) d\eta, \quad \xi \in \mathbb{R},$$

where $\mu_{\pm} = [c \pm \sqrt{c^2 + 4d\gamma}](2d)^{-1}$. Thus, $\overline{w} = \overline{t}U' + \psi \gg 0$, which violates the definition of \overline{t} , and the contradiction yields the conclusion we need. Next, we show that $\mathcal{N}(\Pi_L)^2 = \mathcal{N}(\Pi_L)$, we argue by contradictions. Let $\Pi_L \Phi = t_1 U'$ for some $\Phi \in L^p$ and $t_1 \in \mathbb{R}$, that is, $t_1 U' \in \mathcal{R}(\Pi_L)$. Therefore, $t_1 \int_{\mathbb{R}} \Psi(\eta) U'(\eta) d\eta = 0$, which is a contradiction. with the same reasoning, we can show that $\mathcal{N}(\Pi_L^*) = \text{span}\{\Psi\}$ and 0 is also a simple eigenvalue of Π_L^* . Thus, we proved that (iii) and $\dim \mathcal{N}(\Pi_L) = \dim \mathcal{N}(\Pi_L^*) = \text{codim} \mathcal{R}(\Pi_L) = \text{codim} \mathcal{R}(\Pi_L^*) = 1$. Note that (3.9) is ensured by Lemma 4.2 if Π_L is considered in L^p with $1 \leq p < \infty$. In case $p = \infty$, $\mathcal{R}(\Pi_L) \subseteq \{h \in L^\infty \mid \int h \Psi = 0\}$ implies (3.9). Hence (iv) is completed. Certainly, $\lambda \in \rho(\Pi_L)$ for any $\lambda \in \mathbb{C}$ with $\text{Re} \lambda > 0$. Moreover, by using the arguments similar to those in [3] (see pg 124, also refer to [21]), we can show that $\Pi_L - i\eta$ is injective for any $\eta \in \mathbb{R}$. On the other hand, for each $\eta \in \mathbb{R}$, $\Pi_L - i\eta$ is Fredholm of index zero. Hence $i\eta \in \rho(\Pi_L)$ for any $\eta \in \mathbb{R}$ and (v) is proved. \square

4. APPENDIX

Let $\Delta_{L_{\pm} - \lambda I} : \mathbb{C} \rightarrow \mathbb{C}$ be the characteristic equations associated with the operators $\Pi_{L_{\pm}} - \lambda I$, which is defined by

$$dz^2 - cz - \lambda + a^{\pm} + b^{\pm} \int_{\mathbb{R}} J(s) e^{-zs} ds.$$

Let $\Delta_{L_{\pm} - \overline{\lambda} I}^* : \mathbb{C} \rightarrow \mathbb{C}$ be the characteristic equations associated with the adjoint operators $(\Pi_{L_{\pm}} - \lambda I)^*$, which is defined by

$$dz^2 + cz - \overline{\lambda} + a^{\pm} + b^{\pm} \int_{\mathbb{R}} J(s) e^{-zs} ds.$$

Remark 4.1. *In light of proposition 2.1, it is clear that there exist $\Lambda > 0$ such that $\Delta_{L_+ - \lambda}(z) = 0$ ($\text{res } \Delta_{L_- - \lambda}(z) = 0$) has no solution in the vertical strip $\{\lambda \in \mathbb{C} \mid -\Lambda \leq \text{Re} z \leq \Lambda\}$ provided $\Delta_{L_+ - \lambda}(i\eta) \neq 0$ ($\text{res } \Delta_{L_- - \lambda}(i\eta) \neq 0$) for any $\eta \in \mathbb{R}$. In fact, for any $K > 0$, the set $\{-K \leq \text{Re} z \leq K \mid \Delta_{L_{\pm} - \lambda}(z) = 0\}$ is bounded. Since $\Delta_{L_{\pm} - \lambda}(z)$ is analytic on \mathbb{C} , there are only finite zeros of $\Delta_{L_{\pm} - \lambda}(z)$ located in the strip $-K \leq \text{Re} z \leq K$, thus, there must exist $\Lambda > 0$ such that $\Delta_{L_{\pm} - \lambda}(z) = 0$ has no solution in the vertical strip $\{\lambda \in \mathbb{C} \mid -\Lambda \leq \text{Re} z \leq \Lambda\}$. In addition, if $\Delta_{L_+ - \lambda I}(i\eta) \neq 0$ ($\text{res } \Delta_{L_- - \lambda I}(i\eta) \neq 0$) for any $\eta \in \mathbb{R}$, then $\Delta_{L_+ - \overline{\lambda} I}^*(i\eta) \neq 0$ ($\text{res } \Delta_{L_- - \overline{\lambda} I}^*(i\eta) \neq 0$) for any $\eta \in \mathbb{R}$.*

Definition 4.1. *The operator $\Pi_{L_+} - \lambda I$ ($\Pi_{L_-} - \lambda I$) is called hyperbolic if $\Delta_{L_+ - \lambda I}(i\eta) \neq 0$ ($\Delta_{L_- - \lambda I}(i\eta) \neq 0$) for any $\eta \in \mathbb{R}$. The operator $\Pi_L - \lambda I$ is called asymptotic hyperbolic if both $\Pi_{L_+} - \lambda I$ and $\Pi_{L_-} - \lambda I$ are hyperbolic. Similarly, the operator $(\Pi_{L_+} - \lambda I)^*$ ($(\Pi_{L_-} - \lambda I)^*$) is called hyperbolic if $\Delta_{L_+ - \lambda I}^*(i\eta) \neq 0$ ($\Delta_{L_- - \lambda I}^*(i\eta) \neq 0$) for any $\eta \in \mathbb{R}$. The operator $(\Pi_L - \lambda I)^*$ is called asymptotic hyperbolic if both $(\Pi_{L_+} - \lambda I)^*$ and $(\Pi_{L_-} - \lambda I)^*$ are hyperbolic.*

Proposition 4.1. *If $\lambda \in \mathbb{C}$ such that $\Pi_{L_+} - \lambda I$ is hyperbolic, then the operator $\Pi_{L_+} - \lambda I$ is an isomorphism from $W^{1,p}$ onto L^p for $1 \leq p \leq \infty$ provided $d = 0$. If $d > 0$, then $\Pi_{L_+} - \lambda I$ is an isomorphism from $W^{2,p}$ onto L^p for $1 \leq p \leq \infty$. In each case, the inverse is given by convolution*

$$[(\Pi_{L_+} - \lambda I)^{-1}h](\xi) = (G_{L_+}^\lambda * h)(\xi) = \int_{\mathbb{R}} G_{L_+}^\lambda(\xi - \eta)H(\eta)d\eta$$

with a function $G_{L_+}^\lambda$ which enjoys the estimate

$$(4.1) \quad |G_{L_+}^\lambda(\xi)| \leq K'e^{-\alpha|\xi|}, \quad \xi \in \mathbb{R},$$

for some constants k' and α . Moreover, the same assertion is valid for $\Pi_{L_-} - \lambda I$.

Proof. Invoking Lemma 2.2, we only need to show (4.1). By the remark 4.1, there exist $m > 0$ such that all zeros of $\Delta_{L_+ - \lambda}$ lie outside of the strip $\{\lambda \in \mathbb{C} | |\operatorname{Re} z| \leq m\}$. We define

$$k^+ = \inf\{\operatorname{Re} z : \Delta_{L_+ - \lambda}(z) = 0, \operatorname{Re} z > 0\}$$

and

$$k^- = \sup\{\operatorname{Re} z : \Delta_{L_+ - \lambda}(z) = 0, \operatorname{Re} z < 0\}.$$

Choose $\varepsilon' > 0$ sufficiently small such that $\Delta_{L_+ - \lambda}(z)$ only has finite number of zeros in the strip $k_- - \varepsilon' < \operatorname{Re} z \leq \varepsilon'$ and $\Delta_{L_+ - \lambda}(z)$ is analytic on $\operatorname{Re} z = k_- - \varepsilon'$. Again, we let

$$G_{L_+}^\lambda(\xi) = \int_{\mathbb{R}} e^{i\xi\eta} \Delta_{L_+ - \lambda}^{-1}(i\eta) d\eta.$$

By the reasoning used in the proof of Lemma 2.2, we find, for any $\xi \geq 0$,

$$\begin{aligned} G_{L_+}^\lambda(\xi) &= \sum \operatorname{Res}(e^{z\xi} \Delta_{L_+ - \lambda}^{-1}(z))|_{k_- - \varepsilon' < \operatorname{Re} z \leq \varepsilon'} + e^{(k_- - \varepsilon')\xi} \int_{\mathbb{R}} e^{i\xi\eta} \Delta_{L_+ - \lambda}^{-1}(k_- - \varepsilon' + i\eta) d\eta \\ &= O(\xi^i e^{k_- \xi}), \text{ as } \xi \rightarrow \infty. \end{aligned}$$

Here we assume that the zero with $\operatorname{Re} z = k_-$ is a i th zero of $\Delta_{L_+ - \lambda}(z)$ in the strip $k_- - \varepsilon' < \operatorname{Re} z \leq \varepsilon'$. Analogously, we have

$$G_{L_+}^\lambda(\xi) = O(\xi^j e^{k_+ \xi}), \text{ as } \xi \rightarrow -\infty$$

for some $j \geq 0$. Let $0 < \alpha < \min\{|k_-|, |k_+|\}$, then (4.1) follows. \square

Lemma 4.1. *Assume that $\lambda \in \mathbb{C}$ such that $\Pi_L - \lambda I$ is asymptotic hyperbolic. Suppose $(\Pi_L - \lambda I)u = h$, where $h \in L^p$. $u \in W^{1,p}$ when $d = 0$, and $u \in W^{2,p}$ when $d \neq 0$. Then*

$$(4.2) \quad |u(\xi)| \leq K_1 e^{-\mu|\xi|} \|u\|_{L^\infty} + K_2 \int_{\mathbb{R}} e^{-\mu|\xi - \eta|} |h(\eta)| d\eta, \quad \xi \in \mathbb{R}.$$

$$(4.3) \quad \|u\|_{W^{1,p}} \leq K_3 \|u\|_{L^p} + K_4 \|h\|_{L^p}, \quad \text{if } d = 0.$$

$$(4.4) \quad \|u\|_{W^{2,p}} \leq K_5 \|u\|_{L^p} + K_6 \|h\|_{L^p}, \quad \text{if } d > 0.$$

Here all the constants μ and $K_i (i = 1, \dots, 6)$ are positive and independent of u and h .

Proof. Let $G_{L_+}^\lambda$ and $G_{L_-}^\lambda$ are Green functions for $\Pi_{L_+} - \lambda I$ and $\Pi_{L_-} - \lambda I$, respectively. Invoking (4.1), we may assume that both $G_{L_+}^\lambda$ and $G_{L_-}^\lambda$ enjoy the estimate (4.1). Note that $\Pi_L - \lambda I = \Pi_{L_\pm} - \lambda I + M_\pm$, where M_\pm are given by (??).

Set

$$\begin{aligned} (\tilde{L}_\pm^\lambda u)(\xi) &:= (L_\pm - \lambda I)u(\xi) + (\vartheta_\pm^\lambda M_\pm)(\xi)u(\xi), \\ (M_\pm^\lambda u)(\xi) &:= ((1 - \vartheta_\pm^\lambda)M_\pm)(\xi)u(\xi), \end{aligned}$$

where $\vartheta_\pm^\lambda(\xi)$ are the unit step functions similar to (2.19). Then $(\Pi_L - \lambda I)u = h$ is equivalent to

$$(4.5) \quad \tilde{\Pi}_{L_\pm}^\lambda u = h - M_\pm^\lambda u,$$

where $\tilde{\Pi}_{L_\pm}^\lambda = du'' - cu' + (\tilde{L}_\pm^\lambda u)$. Choose ϑ_\pm^λ such that $\vartheta_\pm^\lambda M_\pm(\xi)$ satisfy the conditions of Proposition 2.1, here we may assume that ϑ_\pm^λ have their jump points at $\pm\sigma$ respectively. Then $\tilde{\Pi}_{L_\pm}^\lambda$ are isomorphisms from $W^{1,p}(W^{2,p})$ onto L^p for $1 \leq p \leq \infty$. Let $\tilde{G}_{L_\pm}^\lambda$ be the Green functions for $\tilde{\Pi}_{L_\pm}^\lambda$, then Proposition 2.1 yields

$$|\tilde{G}_{L_\pm}^\lambda(\xi)| \leq C_\lambda e^{-\mu|\xi|}$$

for some positive constants C_λ, μ . Consequently, we have either

$$\begin{aligned} u(\xi) &= \int_{\mathbb{R}} \tilde{G}_{L_+}^\lambda(\xi, \eta)[-M_+^\lambda u(\eta)]d\eta + \int_{\mathbb{R}} \tilde{G}_{L_+}^\lambda(\xi, \eta)h(\eta)d\eta \\ &= \int_{-\infty}^{\sigma} \tilde{G}_{L_+}^\lambda(\xi, \eta)[-M_+^\lambda u(\eta)]d\eta + \int_{\mathbb{R}} \tilde{G}_{L_+}^\lambda(\xi, \eta)h(\eta)d\eta \\ &\leq \int_{-\infty}^{\sigma} e^{-\mu|\xi-\eta|} \|M_+^\lambda\| \|u\|_{L^\infty} d\eta + \int_{-\infty}^{\infty} e^{-\mu|\xi-\eta|} h(\eta) d\eta \end{aligned}$$

or

$$\begin{aligned} u(\xi) &= \int_{\mathbb{R}} \tilde{G}_{L_-}^\lambda(\xi, \eta)[-M_-^\lambda u(\eta)]d\eta + \int_{\mathbb{R}} \tilde{G}_{L_-}^\lambda(\xi, \eta)h(\eta)d\eta \\ &\leq \int_{-\sigma}^{\infty} e^{-\mu|\xi-\eta|} \|M_-^\lambda\| \|u\|_{L^\infty} d\eta + \int_{-\infty}^{\infty} e^{-\mu|\xi-\eta|} h(\eta) d\eta. \end{aligned}$$

Thus, (4.2) follows. Now we define

$$(4.6) \quad (\Lambda_\omega v)(\xi) = dv''(\xi) - cv'(\xi) - \omega v(\xi),$$

then we have

$$(4.7) \quad \Lambda_\omega u = (\lambda - \omega)u - Lu + h,$$

where $(Lv)(\xi) := L(\xi)v(\xi)$. As long as $\omega > 0$ is sufficiently large $\Lambda_\omega^{-1} : L^p \rightarrow W^{1,p}(W^{2,p})$ exists and satisfies $\|\Lambda_\omega^{-1}u\|_{W^{1,p}} \leq \omega_p \|u\|_{L^p}$ provided $d = 0$. ($\|\Lambda_\omega^{-1}u\|_{W^{1,p}} \leq \omega_p \|u\|_{L^p}$ if $d > 0$), where ω_p only depends on d, c, ω and p . Since L is a bounded operator in L^p , the desired conclusions (4.3) and (4.4) follow. \square

Remark 4.2. In virtue of remark 4.1 and Lemma 4.1, $(\Pi_L - \lambda I)^*$ is asymptotically hyperbolic if and only if $\Pi_L - \lambda I$ is asymptotically hyperbolic, where $\lambda \in \mathbb{C}$. Suppose that $\Pi_L - \lambda I$ is asymptotically hyperbolic and $\mathcal{N}(\Pi_L - \lambda I)$ is nonempty, where $\mathcal{N}(\Pi_L - \lambda I)$ is the kernel of $\Pi_L - \lambda I$. Let $\phi \in \mathcal{N}(\Pi_L - \lambda I)$, then Lemma 4.1 implies that ϕ decays exponentially at infinity. Clearly, the same conclusion holds true for $(\Pi_L - \lambda I)^*$ provided it has nonempty kernel.

Proposition 4.2. Assume that $\lambda \in \mathbb{C}$ such that $(\Pi_L - \lambda I)^*$ is asymptotically hyperbolic. Suppose for some p that there are bounded sequences $u_n \in W^{1,p}$ ($W^{2,p}$ when $d > 0$) and $h_n \in L^p$ such that $(\Pi_L - \lambda I)^* u_n = h_n$ and $h_n \rightarrow \bar{h}$ in L^p . Then there exists a subsequence $u_{n'}$ and some $\bar{u} \in W^{1,p}$ ($W^{2,p}$) such that $u_{n'} \rightarrow \bar{u}$ in $W^{1,p}$ ($W^{2,p}$) and $(\Pi_L - \lambda I)\bar{u} = \bar{h}$.

Proof. Let the operator $N_{\pm}(\xi) : L^p \rightarrow L^p$ defined by

$$N_{\pm}(\xi)u(\xi) = [f_s(U(\xi), J*U(\xi)) - f_s(\pm 1, \pm 1)]u(\xi) + \int_{\mathbb{R}} J(\xi - \eta)[f_s(U(\eta), J*U(\eta)) - f_s(\pm 1, \pm 1)]u(\eta)d\eta.$$

Also set

$$\begin{aligned} (\hat{L}_{\pm}^{\lambda}u)(\xi) &:= (L_{\pm} - \lambda I)u(\xi) + N_{\pm}(\xi)(\vartheta_{\pm}^{\lambda}u)(\xi), \\ (N_{\pm}^{\lambda}u)(\xi) &:= N_{\pm}(\xi)((1 - \vartheta_{\pm}^{\lambda})u)(\xi), \end{aligned}$$

where $\vartheta_{\pm}^{\lambda}(\xi)$ are same as these defined in Lemma 4.1. Let $\hat{\Pi}_{L\pm}^{\lambda} = du'' - cu' + (\hat{L}_{\pm}^{\lambda}u)$. Then, with the same reasoning, we draw the desired conclusion. □

Lemma 4.2. Suppose that $\Pi_L - \lambda I$ is asymptotically hyperbolic. Then the operator $\Pi_L - \lambda I : L^p \rightarrow L^p$ is Fredholm operator for each p ($1 \leq p < \infty$). Furthermore, the range $\mathcal{R}(\Pi_L - \lambda I)$ is given by

$$\mathcal{R}(\Pi_L - \lambda I) = \{h \in L^p \mid \int_{\mathbb{R}} \bar{w}(\xi)h(\xi)d\xi = 0, \quad w(\xi) \in \mathcal{N}((\Pi_L - \lambda I)^*)\}.$$

In particular,

$$\begin{aligned} \dim \mathcal{N}(\Pi_L - \lambda I)^* &= \text{codim} \mathcal{R}(\Pi_L - \lambda I), \quad \dim \mathcal{N}(\Pi_L - \lambda I) = \text{codim} \mathcal{R}(\Pi_L - \lambda I)^*, \\ \text{Ind}(\Pi_L - \lambda I) &= -\text{Ind}(\Pi_L - \lambda I)^*. \end{aligned}$$

In case $p = \infty$, $\Pi_L - \lambda I$ is semi-Fredholm operator. Additionally, the operator $\Pi_L - \lambda I$ is Fredholm if J has compact support.

Proof. The proof of this lemma is very similar to that of Theorem A in [15], we shall therefore only sketch the proof. As usual, we shall only give the proof for the case that $d = 0$ since the proof for the case that $d > 0$ can be completed analogously. We start to show that the unit ball

$$\mathcal{B} = \{u \in W^{1,p} \mid u \in \mathcal{N}(\Pi_L - \lambda I), \|u\|_{W^{1,p}} \leq 1\}$$

in $\mathcal{N}(\Pi_L - \lambda I) \subset W^{1,p}$ is compact, and hence we can conclude $\dim \mathcal{N}(\Pi_L - \lambda I) < \infty$. It is worth pointing out that $\mathcal{N}(\Pi_L - \lambda I)$ is independent of p . Indeed, this can be inferred from

the remark 4.2. Now, we choose any sequence $u_n \in \mathcal{B}$, then by Proposition 2.2 with $h_n = 0$, there exists a subsequence $u_{n'} \rightarrow u^*$ in $W^{1,p}$ for some u^* with $(\Pi_L - \lambda I)u^* = 0$. Therefore, $u^* \in \mathcal{B}$ and \mathcal{B} is compact.

Next we let p be fixed and we show that $\mathcal{R}(\Pi_L - \lambda I)$ is closed. Let $h_n \in \mathcal{R}(\Pi_L - \lambda I) \subseteq L^p$ such that $h_n \rightarrow h^*$ in L^p , then we need to show that $h^* \in \mathcal{R}(\Pi_L - \lambda I)$. Let $\mathcal{C} \subseteq W^{1,p}$ be a closed subspace complement of $\mathcal{N}(\Pi_L - \lambda I)$, that is, $W^{1,p} = \mathcal{N}(\Pi_L - \lambda I) \oplus \mathcal{C}$. Clearly, there exists a sequence $u_n \in \mathcal{C}$ such that $(\Pi_L - \lambda I)u_n = h_n$. As shown in [15], $\|u_n\|_{W^{1,p}}$ must be bounded, hence Proposition 2.2 implies that there exists $u^* \in \mathcal{C}$ such that $(\Pi_L - \lambda I)u^* = h^*$. This prove the closeness of $\mathcal{R}(\Pi_L - \lambda I)$. Therefore, $\Pi_L - \lambda I$ is semi-Fredholm.

Now, we assume that $1 \leq p < \infty$, in order to prove $(\Pi_L - \lambda I)$ is Fredholm, it suffices to show that $\mathcal{R}(\Pi_L - \lambda I)$ has finite codimensions in L^p . To this end, we let $\mathcal{N}((\Pi_L - \lambda I)^*)^0_p \subseteq L^p$ denote

$$\mathcal{N}((\Pi_L - \lambda I)^*)^0_p = \{h \in L^p \mid \int_{\mathbb{R}} \overline{w}(\xi)h(\xi)d\xi = 0, \quad w(\xi) \in \mathcal{N}((\Pi_L - \lambda I)^*)\}.$$

From remark 4.2, we see that $(\Pi_L - \lambda I)^*$ is also asymptotically hyperbolic, and hence Proposition 2.2 together above arguments imply that $\dim \mathcal{N}((\Pi_L - \lambda I)^*) < \infty$, where $\mathcal{N}((\Pi_L - \lambda I)^*) \subseteq W^{1,q}$. Certainly, $\text{codim} \mathcal{N}((\Pi_L - \lambda I)^*)^0_p = \dim \mathcal{N}((\Pi_L - \lambda I)^*) < \infty$. To complete the proof, we show that $\mathcal{N}((\Pi_L - \lambda I)^*)^0_p = \mathcal{R}(\Pi_L - \lambda I)$. In view of (3.5), we have $\mathcal{R}(\Pi_L - \lambda I) \subseteq \mathcal{N}((\Pi_L - \lambda I)^*)^0_p$. Assume for some p ($1 \leq p < \infty$) that $\mathcal{R}(\Pi_L - \lambda I) \neq \mathcal{N}((\Pi_L - \lambda I)^*)^0_p$, then there exists $v^* \in \mathcal{R}(\Pi_L - \lambda I)^\perp$ and $\int_{\mathbb{R}} \overline{v^*}(\xi)h(\xi)d\xi \neq 0$ for some $h \in \mathcal{N}((\Pi_L - \lambda I)^*)^0_p$, where $\mathcal{R}(\Pi_L - \lambda I)^\perp = \{v \in L^q \mid \int_{\mathbb{R}} \overline{v}(\xi)g(\xi)d\xi = 0, \quad g \in \mathcal{R}(\Pi_L - \lambda I)\}$. Clearly, $v^* \notin \mathcal{N}((\Pi_L - \lambda I)^*)$. On the other hand, we have

$$\int_{\mathbb{R}} \overline{v(\xi)}(\Pi_L - \lambda I)u(\xi)d\xi = 0, u \in W^{1,p}.$$

Choose any $\chi \in C^\infty(\mathbb{R}, \mathbb{C})$ with compact support and set $u = \overline{\chi}$. By taking the complex conjugates, we find

$$\begin{aligned} 0 &= \int_{\mathbb{R}} v(\xi) \overline{(\Pi_L - \lambda I)\chi(\xi)} d\xi \\ &= d \int_{\mathbb{R}} \chi(\xi) v''(\xi) d\xi + c \int_{\mathbb{R}} \chi(\xi) v'(\xi) d\xi + \int_{\mathbb{R}} \chi(\xi) (L^*(\xi) - \overline{\lambda}) v(\xi) d\xi \\ &= \int_{\mathbb{R}} \chi(\xi) ((\Pi_L - \lambda I)^* v)(\xi) d\xi. \end{aligned}$$

This indicates that v solves the adjoint equation in the sense of distributions and $v \in W^{1,q}$. Thus $v \in \mathcal{N}((\Pi_L - \lambda I)^*)$. This contradiction establishes that $\mathcal{R}(\Pi_L - \lambda I) = \mathcal{N}((\Pi_L - \lambda I)^*)^0_p$ for $1 \leq p < \infty$.

For the case of $p = \infty$, once again, (3.5) implies that $\mathcal{R}(\Pi_L - \lambda I) \subseteq \mathcal{N}((\Pi_L - \lambda I)^*)^0_\infty$. Suppose that J has compact support. We need to show that $\mathcal{R}(\Pi_L - \lambda I) \supseteq \mathcal{N}((\Pi_L - \lambda I)^*)^0_\infty$. We start to show that every $h \in \mathcal{N}((\Pi_L - \lambda I)^*)^0_\infty$ can be written as $h = h_1 + h_2$, where $h_1 \in \mathcal{R}(\Pi_L - \lambda I)$, and $h_1 = h$ whenever $|\xi| \geq \tau$ for some positive constant τ . Certainly,

$h_2 \in \mathcal{N}((\Pi_L - \lambda I)^*)_\infty^0$ and h_2 has compact support. Therefore, $h_2 \in \mathcal{N}((\Pi_L - \lambda I)^*)_p^0$. As shown before, $h_2 \in \mathcal{R}(\Pi_L - \lambda I)$. Hence we have $h \in \mathcal{R}(\Pi_L - \lambda I)$ as desired. To construct h_1 , we let $w_\pm = (\tilde{\Pi}_{L_\pm}^\lambda)^{-1}h$, Where $\tilde{\Pi}_{L_\pm}^\lambda$ are isomorphisms given in Lemma 4.1. Due to Lemma 4.1, there exists $\sigma > 0$ such that $(\Pi_L - \lambda I)w_+ = h$ for any $\xi \geq \sigma$, while $(\Pi_L - \lambda I)w_- = h$ for any $\xi \leq -\sigma$. Now, we let $w^*(\xi) = m(\xi)w_+(\xi) + (1 - m(\xi))w_-(\xi)$, where $m : \mathbb{R} \rightarrow \mathbb{R}^+$ is a C^2 function such that $m(\xi) = 0$ for $\xi \leq 0$, and $m(\xi) = 1$ for $\xi \geq 1$. Since J has compact support, the direct computation shows that $(\Pi_L - \lambda I)w^* = h$ provided $|\xi|$ is sufficiently large. Choose $h_1 = (\Pi_L - \lambda I)w^*$, as required. Hence the proof is completed. \square

Now we set $\bar{\tau} = \max\{a^+ + b^+, a^- + b^-\}$, $\underline{\tau} = \min\{a^+ - b^+, a^- - b^-\}$.

$\Omega_+ = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda > \bar{\tau}\}$, $\Omega_- = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda < \underline{\tau}\}$, $\Xi = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda > \bar{\tau}\} \cup \{\lambda \in \mathbb{C} | \operatorname{Im} z| > c^2 \sqrt{\bar{\tau} - \operatorname{Re} z} + (b^+ \wedge b^-)\}$.

Proposition 4.3. *If $\lambda \in \Xi$ then $\lambda I - \Pi_L$ is asymptotic hyperbolic when $d > 0$. In case $d = 0$, the same conclusion also holds true if $\lambda \in \Omega_+ \cup \Omega_-$.*

Proof. It is sufficient to prove that $\Delta_{L_\pm - \lambda}(i\eta) \neq 0$ for any $\eta \in \mathbb{R}$ provided $\operatorname{Re} \lambda > \bar{\tau}$. Note that $\Delta_{L_\pm - \lambda}(i\eta) = 0$ if and only if

$$ic\eta + d\eta^2 + (\lambda - a^\pm) = b^\pm \int_{\mathbb{R}} J(s) e^{-i\eta s} ds.$$

First we note that

$$d\eta^2 + \operatorname{Re} \lambda - a^\pm > b^\pm \geq |b^\pm \int_{\mathbb{R}} J(s) e^{-i\eta s} ds|, \quad \eta \in \mathbb{R}$$

when $\operatorname{Re} \lambda > \bar{\tau}$, hence $\Delta_{L_\pm - \lambda}(i\eta) \neq 0$ for all $\eta \in \mathbb{R}$, provided $\operatorname{Re} \lambda > \bar{\tau}$. Now, suppose $\operatorname{Re} \lambda \leq \bar{\tau}$, it is easy to see that

$$c^{-2}(\operatorname{Im} z - b^\pm \int_{\mathbb{R}} J(s) \sin(-\eta s) ds)^2 + \operatorname{Re} z - a^\pm > b^\pm \int_{\mathbb{R}} J(s) \cos(\eta s) ds, \quad \eta \in \mathbb{R},$$

whenever $|\operatorname{Im} z| > c^2 \sqrt{\bar{\tau} - \operatorname{Re} z} + (b^+ \wedge b^-)$. In case $d = 0$, we still have $\Delta_{L_\pm - \lambda}(i\eta) \neq 0$ if $\operatorname{Re} \lambda > \bar{\tau}$. Moreover, $\operatorname{Re} \lambda < \underline{\tau}$ implies

$$\operatorname{Re} \lambda - a^\pm < -b^\pm \leq \operatorname{Re}(b^\pm \int_{\mathbb{R}} J(s) e^{-i\eta s} ds), \quad \eta \in \mathbb{R}.$$

Therefore, the desired conclusion follows. \square

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